MATH 2050C Lecture 16 (Mar 15)

Recall: "ε-δ def$^{\underline{n}}$ for limit of functions"

$f: A \subseteq \mathbb{R} \to \mathbb{R}$ $\qquad c \in \mathbb{R}$ cluster point

$$\boxed{\lim_{x \to c} f(x) = L} \iff \forall \varepsilon > 0,\ \exists\, \delta = \delta(\varepsilon) > 0 \text{ s.t. } |f(x) - L| < \varepsilon$$

whenever $x \in A$, $0 < |x - c| < \delta$

Example: Use ε-δ def$^{\underline{n}}$ to show

$$\boxed{\lim_{x \to 1} \frac{x^2 - 2}{x + 1} = -\frac{1}{2}}$$

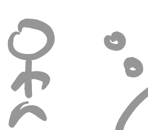
Note: $f: A := \mathbb{R} \setminus \{-1\} \to \mathbb{R}$, $\quad f(x) := \dfrac{x^2 - 2}{x + 1}$.

1 is a cluster pt of $A = \mathbb{R} \setminus \{-1\}$.

Proof: Let $\varepsilon > 0$ be fixed but arbitrary.

[**Want:** Choose $\delta = \delta(\varepsilon) > 0$ s.t.

$$\left.\begin{array}{l} 0 < |x - 1| < \delta \\ A \ni x \neq -1 \end{array}\right\} \Rightarrow \left| \frac{x^2 - 2}{x + 1} - \left(-\frac{1}{2}\right) \right| < \varepsilon$$]



ASSUME: $0 < |x - 1| < \delta$

$$\left|\frac{x^2-2}{x+1}+\frac{1}{2}\right| = \left|\frac{2x^2-4+x+1}{2(x+1)}\right| = \left|\frac{2x^2+x-3}{2(x+1)}\right|$$

$$= \left|\frac{(x-1)(2x+3)}{2(x+1)}\right| = \frac{1}{2}\underbrace{\boxed{\frac{|2x+3|}{|x+1|}}}_{\text{bdd? when } x\approx 1}\ \underbrace{|x-1|}_{\text{"small"}} < \frac{7}{2}\delta \le \varepsilon$$

If $0<|x-1|<1$, then

$$0<x<2 \Rightarrow \begin{matrix} 3<2x+3<7 \\ 1<x+1<3 \end{matrix} \Rightarrow \begin{matrix} |2x+3|<7 \\ |x+1|>1 \end{matrix}$$

Choose $\delta := \min\{1, \frac{2\varepsilon}{7}\} > 0$.

<u>Note</u>: If $|x-1|<\delta\le 1$, then $0<x<2$

Hence, $|2x+3|<7$ & $|x+1|>1$.

$\forall x\in A$, $0<|x-1|<\delta$, we have

$$\left|\frac{x^2-2}{x+1}-\left(-\frac{1}{2}\right)\right| = \left|\frac{2x^2+x-3}{2(x+1)}\right| = \left|\frac{(x-1)(2x+3)}{2(x+1)}\right|$$

$$= \frac{1}{2}\,\frac{|2x+3|}{|x+1|}\cdot|x-1| < \frac{1}{2}\cdot\frac{7}{1}\cdot\delta \le \varepsilon$$

$\square$

Prop: $\lim_{x \to c} f(x)$, if exists, is unique.

Pf: Exercise!

Q: How are the concept of limits for seq. and functions related?

Thm: "Sequential Criteria" $f: A \to \mathbb{R}$

$\boxed{\lim_{x \to c} f(x) = L}$ $\iff$ $\forall$ seq. (x_n) in A s.t.

cluster pt. limit of fcn

$$(*) \begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim(x_n) = c \end{cases}$$

we have $\boxed{\lim(f(x_n)) = L}$

limit of seq.

Proof: "$\Rightarrow$" Assume $\lim_{x \to c} f(x) = L$.

Let (x_n) be any seq in A s.t. $(*)$ holds

Claim: $\lim(f(x_n)) = L$.

Pf: (Check this ε-K def$^{\underline{n}}$).

Let $\varepsilon > 0$ be fixed but arbitrary.

Since $\lim_{x \to c} f(x) = L$, by ε-δ def$^{\underline{n}}$ of limit.

$\exists\ \delta = \delta(\varepsilon) > 0$ s.t

$|f(x) - L| < \varepsilon$ whenever $x \in A$
$0 < |x - c| < \delta$

Since $\lim(x_n) = c$, for the $\delta > 0$ above,

$\exists\ K = K(\delta) \in \mathbb{N}$ s.t. $x_n \in A$

$0 < |x_n - c| < \delta$ when $n \geq K$
(*)

Therefore, $|f(x_n) - L| < \varepsilon$ when $n \geq K$.

"$\Leftarrow$" Argue by contradiction.

Suppose on the contrary, assume the R.H.S. holds

but "$\lim\limits_{x \to c} f(x) \neq L$" (ie f does NOT converge to L)

By taking the negation of the ε-δ def^{n} of limit of fcn, we have: $\exists\ \varepsilon_0 > 0$ s.t. $\forall\ \delta > 0$

$\exists\ x_\delta \in A$ and $0 < |x_\delta - c| < \delta$

BuT: $|f(x_\delta) - L| \geq \varepsilon_0$

Take $\delta := \frac{1}{n}$, $n \in \mathbb{N}$, then get $x_n \in A$

and $0 < |x_n - c| < \frac{1}{n}$ $\quad \forall n \in \mathbb{N}$

and $|f(x_n) - L| \geq \varepsilon_0$ ——— (#)

Consider this seq (x_n) in A, note that

$$(*) \begin{cases} x_n \neq c \quad \forall n \in \mathbb{N} \\ \lim (x_n) = c \end{cases}$$ is satisfied

But: we do **NOT** have $\lim(f(x_n)) = L$

because of (#). A contradiction!

$\square$

Remark: In particular, the seq. criteria is very useful to show $\lim_{x \to c} f(x)$ does **not** exist.

Cor 1:

f DOES NOT converge to L as $x \to c$ $\iff$ $\exists$ seq. (x_n) in A s.t. $\begin{cases} x_n \neq c \quad \forall n \in \mathbb{N} \\ \lim(x_n) = c \end{cases}$

But $(f(x_n)) \not\to L$

Cor 2: "Divergence Criteria"

f "DIVERGES" as $x \to c$ $\iff$ $\exists$ seq. (x_n) in A s.t. $\begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim(x_n) = c \end{cases}$

i.e. f DOES NOT converge to any $L \in \mathbb{R}$ as $x \to c$.

BUT $(f(x_n))$ is divergent.

Proof of Cor 2:

"$\Leftarrow$" Pf: Exercise.

"$\Rightarrow$" Argue by contradiction.

Suppose f diverges as $x \to c$, but the R.H.S. fails to hold. i.e. $\forall$ seq. (x_n) in A s.t.

$$(*) \begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim(x_n) = c \end{cases}$$

we have $(f(x_n))$ must be convergent, so

$$\lim(f(x_n)) = L \text{ for some } L \in \mathbb{R}$$

Caution: This may depend on the choice of (x_n).

Claim: The limit L DOES NOT depend on (x_n)

Pf: Suppose (x_n), (x_n') sastisfying (*), and

$$\lim(f(x_n)) = L, \quad L' = \lim(f(x_n'))\,.$$

Consider the new "approximating seq.":

$$(y_n) := (x_1, x_1', x_2, x_2', x_3, x_3', \dots)$$

then $y_n \neq c \ \forall n \in \mathbb{N}$ and $\lim(y_n) = c$

So, by hypothesis on R.H.S..

$$(f(y_n)) := (f(x_1), f(x_1'), f(x_2), f(x_2'), \dots)$$

L

L'

is CONVERGENT, so $L = L'$ $\square$

By seq criteria, we have $\lim\limits_{x \to c} f(x) = L$. Contradiction.

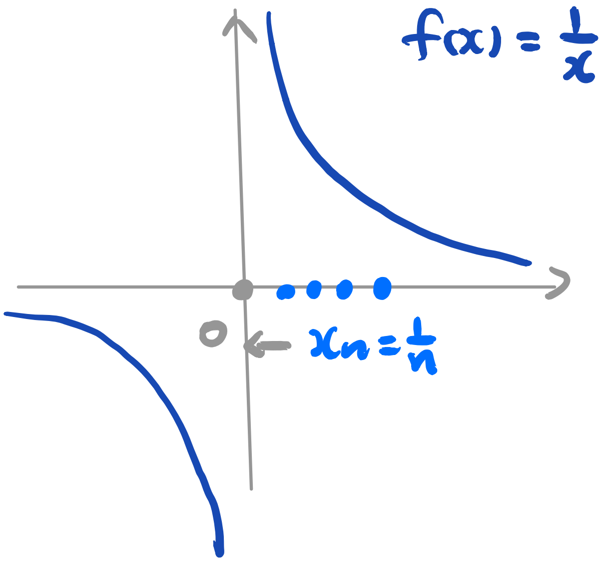
$\square$

Let's look at a few examples.

Example 1: $\lim_{x \to 0} \frac{1}{x}$ does NOT exist.

$f : A = \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$f(x) := 1/x$



Take the seq $(x_n) := \left(\frac{1}{n}\right)$

$\begin{cases} x_n \neq 0 \quad \forall n \in \mathbb{N} \\ \lim(x_n) = 0, \quad x_n \in A \end{cases}$

BUT the image seq

$$(f(x_n)) = (n)$$

is DIVERGENT. ($\because$ unbdd).

By Cor. 2 above, we are done.

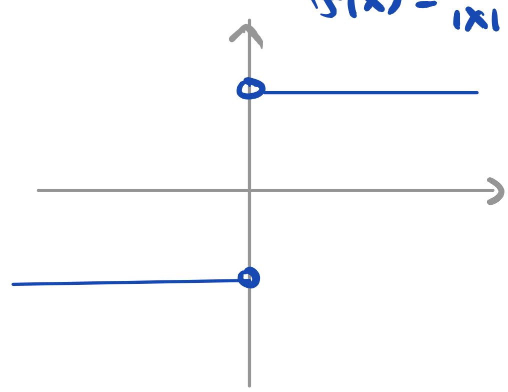
$\square$

Example 2: ("The sign function")

$f : A = \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$ ie $f(x) = \frac{x}{|x|}$

Claim: $\lim_{x \to 0} \frac{x}{|x|}$ does NOT exist!



Pf: Take $(x_n) := \left(\frac{(-1)^n}{n}\right) \to 0$, $\neq 0$

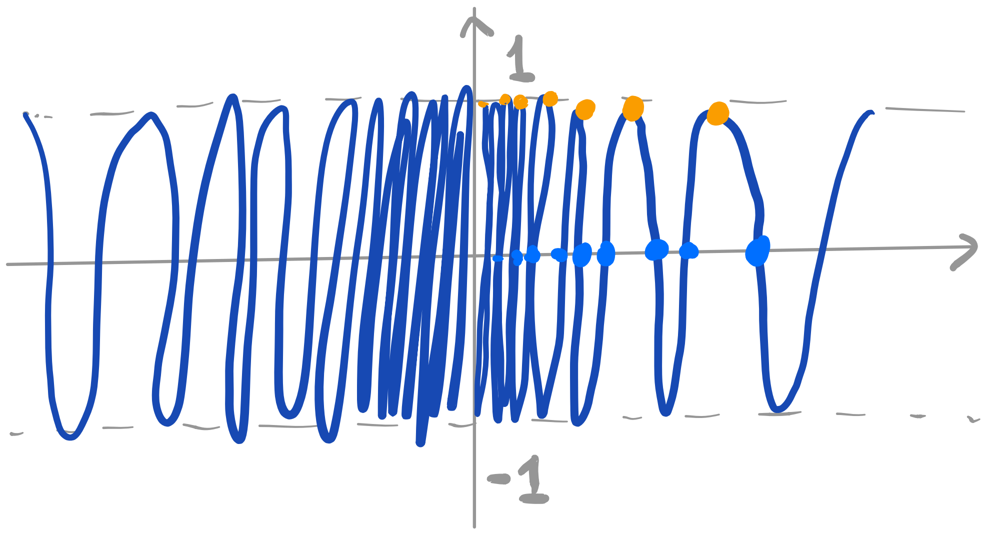
BUT $(f(x_n)) = ((-1)^n)$

is DIVERGENT.

Example 3: $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does NOT exist.

$f: A = \mathbb{R}\setminus\{0\} \to \mathbb{R}$

$f(x) = \sin\frac{1}{x}$



Take $(x_n) := \left(\overset{\neq 0}{\frac{1}{n\pi}}\right) \to 0$ But $(f(x_n)) = (0) \to 0$

take $(x_n') := \left(\overset{\neq 0}{\frac{1}{\frac{\pi}{2}+2n\pi}}\right) \to 0$ But $(f(x_n')) = (1) \to 1$

Choose $(y_n) := (x_1, x_1', x_2, x_2', x_3, x_3', \dots) \to 0$

But: $(f(y_n)) = (0, 1, 0, 1, 0, 1, \dots)$

is DIVERGENT.